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The heat kernel approach for calculating the effective action in quantum field theory and quantum gravity

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Abstract

A short informal overview about recent progress in the calculation of the effective action in quantum gravity is given. I describe briefly the standard heat kernel approach to the calculation of the effective action and discuss the applicability of the Schwinger - De Witt asymptotic expansion in the case of strong background fields. I propose a new ansatz for the heat kernel that generalizes the Schwinger - De Witt one and is always valid. Then I discuss the general structure of the asymptotic expansion and put forward some approximate explicitly covariant methods for calculating the heat kernel, namely, the high-energy approximation as well as the low-energy one. In both cases the explicit formulae for the heat kernel are given.

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First of all I would like to thank Dr. M. Basler and Prof. Kramer for their kind invitation and hospitality expressed to me in the University of Jena. It is a great pleasure for me to present this talk here in Jena where a lot of first class research in gravitation has been done.

However, I will not concern the more or less well studied classical gravity but will try to give an overview about some recent research in *quantum* gravity. More precisely, I will describe some methods of calculations that turned out to be very effective and powerful in quantum field theory and gauge theories.

I start with some introductory notes on quantum field theory, namely, the *formal definition* of the generating function and that of the effective action. Then I will try to explain how this formal definition becomes meaningful in *perturbation theory* after *regularization*. The most convenient way to do this is to employ the *heat kernel* approach and the ζ-function regularization. In what follows I will talk mainly about the various approximations used in covariant calculations of the effective action, namely the *Schwinger - De Witt asymptotic expansion* as well as so called *high-energy* approximation and the *low-energy* one.

1. Formal scheme of quantum field theory

So, let us consider first some set of fields $\varphi^i \equiv \varphi^A(x) = \{\varphi^a(x), \psi^i(x), \mathcal{B}^c_\mu, h_{\mu\nu}, ...\}$ defined on some, say, asymptotic flat Riemannian manifold M with the metric of the Minkovski signature $(- + \cdots +)$. The classical dynamics of these fields is described by the classical equations of motion

$$S_{,i} \equiv \frac{\delta S}{\delta \varphi^A(x)} = 0 \tag{1}$$

derived from the classical action functional

$$S(\phi) = \int_M \mathcal{L}\left(\varphi(x), \partial_\mu \varphi(x)\right) \tag{2}$$

.

After the quantization the fields become field operators $\hat{\varphi}$ acting on some state vectors of a Hilbert space, so called Fock space. Let us suppose that there exist some initial $|in>$ and final $< out|$ vacuum states defined in some appropriate way. At least in asymptotically flat manifolds this can be done consistently.

What is really important is that the vacuum-vacuum transition amplitude can be presented as a Feynman path integral

$$< out|in > \equiv Z(J) = \int \mathcal{D}\varphi \exp\left\{ i\left(S(\varphi) + J_i \varphi^i\right)\right\} \tag{3}$$

Here we introduced the classical sources J_i to investigate the linear reaction of the system on the external perturbation. All the content of quantum field theory with all quantum effects is contained in the following Green functions

$$\left\langle \hat{\varphi}^{i_1} \cdots \hat{\varphi}^{i_n} \right\rangle \equiv \frac{\left\langle out \right| T\left(\hat{\varphi}^{i_1} \cdots \hat{\varphi}^{i_n}\right) \left| in > \right.}{< out|in >} \tag{4}$$

where T means the chronological ordering operator, i.e. the fields must be arranged from left to right in order of decreasing time arguments.

It is easy to show that all these Green functions can be obtained by the functional differentiation of the functional $Z(J)$, that is called, therefore, the generating functional. Moreover, factorizing the unconnected contributions one comes to generating functional for connected Green functions $W(J) = -i \log Z(J)$

$$\left\langle \hat{\varphi}^{i_1} \cdots \hat{\varphi}^{i_n} \right\rangle = i^{-n} \exp(-iW) \frac{\delta^n}{\delta J_{i_1} \cdots \delta J_{i_n}} \exp(iW). \tag{5}$$

The lowest connected Green functions have special names: the mean field

$$< \varphi^i > \equiv \phi^i(J) = \frac{\delta W}{\delta J_i} \tag{6}$$

and the propagator

$$\left\langle \left(\hat{\varphi}^i - \phi^i\right) \left(\hat{\varphi}^k - \phi^k\right) \right\rangle \equiv -i \mathcal{G}^{ik}(J) = -i \frac{\delta^2 W}{\delta J_i \delta J_k} \tag{7}$$

Further, the connected Green functions are expressed in terms of vertex functions. The generating functional for vertex functions is defined then by the Legendre functional transform

$$\Gamma(\phi) = W(J(\phi)) - J_k(\phi)\phi^k \tag{8}$$

This is the most important object in quantum field theory. It contains all the information about the quantized fields.

1) First, one can show that it satisfies the equation

$$\Gamma_{,i}(\phi) \equiv \frac{\delta \Gamma}{\delta \phi^i} = -J_i(\phi) \tag{9}$$

and, therefore, gives the effective equations for the mean field. These equations replace the classical equations of motion and describe the effective dynamics of background fields taking into account all quantum corrections! Thus $\Gamma(\Phi)$ is called usually the *effective action*.

2) Second, it determines the full or exact propagator of quantized fields

$$-\Gamma_{,ik} \mathcal{G}^{kn} = \delta^n_i \tag{10}$$

where $\delta^n_i \equiv \delta^B_A \delta(x - y)$, and the vertex functions

$$\Gamma_n \equiv \Gamma_{,i_1 \cdots i_n}, \qquad (n \geq 3) \tag{11}$$

This means that any S-matrix amplitude, or any Green function, is expressed in terms of propagator and vertex functions that are determined by the effective action.

3) At last, when the test sources vanish the effective action is just the vacuum amplitude

$$< out|in > \Big|_{J=0} = \exp{(i\Gamma)}\,\big|_{J=0} \tag{12}$$

It determines then the probability that the *out*-vacuum is still a vacuum and does not contain *in*-particles

$$| < out|in > |^2 \Big|_{J=0} = \exp{(-2\mathrm{Im}\Gamma)}\,\big|_{J=0} \tag{13}$$

Therefore, the imaginary part of the effective action determines the total number of created particles.

2. Perturbation theory

Let us rewrite the definition of the effective action. To give more sense to the path integral it is convenient to make a so called Wick rotation, or Euclidization, i.e. one replaces the real time coordinate to the purely imaginary one $x^0 \to ix^0$ and singles out the imaginary factor also from the action $S \to iS$ and the effective action $\Gamma \to i\Gamma$. Then the metric of the Riemannian manifold becomes Euclidean, i.e. positive definite, and the classical action in all *good* field theories becomes positive definite functional. So, the Euclidean effective action is defined to satisfy the equation

$$\exp\left(-\frac{1}{\hbar}\Gamma(\phi)\right) = \int \mathcal{D}\varphi \exp\left\{-\frac{1}{\hbar}\left(S(\varphi) - (\varphi^i - \phi^i)\Gamma_{,i}(\phi)\right)\right\} \tag{14}$$

This path integral is still not well defined. There is not any reasonable method, except for lattice theories, to calculate this integral in general case. The only path integrals that can be well defined are the Gaussian ones

$$\int \mathcal{D}h \exp\left\{-\frac{1}{2}\left(h^i F_{ik} h^k\right)\right\} = \mathcal{N}(\mathrm{Det}F)^{-1/2} \tag{15}$$

$$\frac{\int \mathcal{D}h \exp\left\{-\frac{1}{2}\left(h^i F_{ik} h^k\right)\right\} h^{j_1}\cdots h^{j_n}}{\int \mathcal{D}h \exp\left\{-\frac{1}{2}\left(h^i F_{ik} h^k\right)\right\}} = \begin{cases} \frac{(2m)!}{m!} G^{(j_1 j_2} \cdots G^{j_{2m-1} j_{2m})}, & \text{n=2m} \\ 0 & \text{n=2m+1} \end{cases} \tag{16}$$

Here $\mathrm{Det}F$ is the functional determinant of the operator F, $\mathcal{N}$ is some inessential, actually infinite, constant that could be put, in principle equal to 1, and $G^{ik} = (F_{ik})^{-1}$ is the Green function of the operator F with Euclidean boundary conditions, i.e it must be regular and bounded at the infinity. Thus the full path integral can be well defined as an asymptotic series of Gaussian ones. This is just the quasiclassical, or WKB, approximation in the usual quantum mechanics. We decompose the fields in the classical and quantum parts

$$\varphi = \phi + \sqrt{\hbar} h \tag{17}$$

and look for a solution of the equation for the effective action in form of an asymptotic series in powers of Planck constant.

$$\Gamma(\phi) = S(\phi) + \sum_{n\geq 1} \hbar^n \Gamma_{(n)}(\phi) \tag{18}$$

Then all the coefficients of this expansion can be found. They are expressed in terms of the well-known Feynman diagrams. The number of loops in these diagrams correspond to the power of the Planck constant. We will be interested below in the so called *one-loop effective action*

$$\Gamma_{(1)} = \frac{1}{2} \log \mathrm{Det} F \tag{19}$$

where $F_{ik} = S_{,ik}$.

3. ζ-function regularization

Although this quantity seems to be very easy it is still ill defined. The point is it is divergent. This is just the well-known ultraviolet divergence of the quantum field theory. Indeed, we can rewrite the functional determinant as

$$\Gamma_{(1)} = \frac{1}{2} \log \prod_n \lambda_n = \frac{1}{2} \sum_n \log \lambda_n \tag{20}$$

where λ_n are the eigenvalues of the operator F. This series is easy to show to be divergent.

That means that one needs a regularization. This point was investigated very thoroughly by many authors and it is found that in quantum gravity and gauge theories the most appropriate regularizations are the analytical ones. The functional determinants can be well defined in terms of the so called ζ-function. It is defined by

$$\zeta(p) = \mu^{2p} \mathrm{Tr} F^{-p} = \frac{\mu^{2p}}{\Gamma(p)} \int\limits_0^\infty dt t^{p-1} \mathrm{Tr} U(t) \tag{21}$$

Here $U(t)$ is the so called *heat kernel* and $\mathrm{Tr}U(t)$ is the functional trace of it

$$\mathrm{Tr} U(t) = \int dx \mathrm{tr} U(t) \Big|_{diag} \tag{22}$$

$$U(t)\Big|_{diag} = \exp(-tF)\delta(x, x')\Big|_{x=x'} \tag{23}$$

It is easy to show that the integral over t converge for sufficiently large Rep. In the rest of the complex plane of p the ζ-function should be defined by analytic continuation. This analytic continuation leads then to a meromorphic function with some poles on real axis. But the most important point is that it is analytic at the point $p = 0$. This means

that one can calculate the values of the ζ-function and its derivatives at the point $p = 0$. Formally $\zeta(0)$ is equal to the total number of modes of the operator F

$$\zeta(0) = \mathrm{Tr}1 = \sum_n 1 \tag{24}$$

i.e. one can say that $\zeta(0)$ counts the modes of the operator F.

It is almost obvious that the formal derivative of the ζ-function is equal to

$$\zeta'(0) = -\mathrm{Tr}\log\frac{F}{\mu^2} = -\log\mathrm{Det}\frac{F}{\mu^2} \tag{25}$$

where $\zeta'(p) = d\zeta(p)/dp$, i.e. the functional determinant of the operator F. Here μ is a dimensionful parameter which is introduced to preserve dimensions. It is called usually renormparameter.

So, the one-loop effective action becomes now a well defined object

$$\Gamma_{(1)} = -\frac{1}{2}\zeta'(0) \tag{26}$$

Moreover, studying the change of the renormparameter

$$\mu\frac{\partial}{\mu}\zeta(p) = 2p\zeta(p) \tag{27}$$

$$\mu\frac{\partial}{\mu}\Gamma_{(1)} = -\zeta(0) \tag{28}$$

one can get the explicit dependence of the effective action on μ

$$\Gamma_{(1)}(\mu) = -\zeta(0)\log\frac{\mu}{M} + \Gamma_{(1)}(M) \tag{29}$$

where M is some fixed mass or energy parameter which cab be chosen from some physical grounds.

4. Heat kernel for operators of Laplace type

Everything said so far was rather formal. Now we will be more concrete. As you have seen the most important thing in one-loop calculations is the heat kernel. For a very wide class of field theories it is sufficient to consider the second order differential operator of Laplace type, namely

$$F = -\Box + Q + m^2 \tag{30}$$

where

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = g^{-1/2}(\partial_\mu + \mathcal{A}_\mu)g^{1/2}g^{\mu\nu}(\partial_\nu + \mathcal{A}_\nu) \tag{31}$$

is the Laplacian (or Dalambertian in hyperbolic case), $Q = \{Q^A{}_B(x)\}$ is a matrix-valued potential term, $\mathcal{A}_\mu = \{\mathcal{A}^A{}_{B\mu}(x)\}$ is an arbitrary connection and m is a mass parameter. Every second order operator with leading symbol given by metric tensor can be put in this way.

So, the heat kernel is defined now by requiring it to satisfy the heat equation

$$\left(\frac{\partial}{\partial t} + F\right) U(t|x, x') = 0 \tag{32}$$

with the following initial condition

$$U(0|x, x') = g^{-1/2}(x)\delta(x, x') \tag{33}$$

To solve this equation we single out first the asymptotic factor

$$U(t|x, x') = (4\pi t)^{-d/2} \Delta^{1/2}(x, x') \exp\left(-\frac{\sigma(x, x')}{2t}\right) \mathcal{P}(x, x')\Omega(t|x, x') \tag{34}$$

Here $\sigma(x, x')$ is the geodetic interval defined as one half the square of the length of the geodesic between points x and x', $\Delta(x, x')$ is the Van Vleck - Morette determinant and $\mathcal{P}(x, x')$ is the parallel displacement operator. The trace of the heat kernel that defines ζ-function and the effective action is expressed in terms of the transfer function $\Omega(t)$

$$\mathrm{Tr}U(t) = (4\pi t)^{-d/2} \mathrm{Tr}\Omega(t) \tag{35}$$

which satisfies the equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{t}D + \bar{F}\right) \Omega(t|x, x') = 0 \tag{36}$$

with initial condition

$$\Omega(0|x, x')\Big|_{x=x'} = 1 \tag{37}$$

where $D = \nabla_\mu \sigma^\mu$ is the differential operator along the geodesic and $\bar{F}$ is defined by

$$\bar{F} = \mathcal{P}^{-1}\Delta^{-1/2} F \Delta^{1/2}\mathcal{P} = -\mathcal{P}^{-1}\Delta^{-1/2} \,\Box\, \Delta^{1/2}\mathcal{P} + \mathcal{P}^{-1}Q\mathcal{P} + m^2 \tag{38}$$

The usual way to solve this equation is to employ so called Schwinger - De Witt ansatz De Witt (1965)

$$\Omega(t) = \exp(-tm^2) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k \tag{39}$$

where

$$b_n = \sum_{k=0}^{n} \binom{n}{k} m^{2(n-k)} a_k \tag{40}$$

Then one gets the well known De Witt recursion relation for the coefficients a_k

$$\left(1+\frac{1}{k}D\right)a_k=\bar F_0 a_{k-1} \tag{41}$$

where

$$\bar F_0=\bar F\big|_{m=0}=-\mathcal{P}^{-1}\Delta^{-1/2}\,\Box\,\Delta^{1/2}\mathcal{P}+\mathcal{P}^{-1}Q\mathcal{P} \tag{42}$$

These coefficients play very important role both in physics and mathematics. We call them Hadamard - Minackshisundaram - De Witt - Seeley (HMDS) coefficients. The calculation of these coefficients in general case of arbitrary background is in itself of great importance and offers a complicated technical problem. Up to now only four lowest order coefficients, more precisely their coincidence limits were calculated. The pioneering method of De Witt (1965) is quite simple but gets very cumbersome at higher orders. By means of it only two coefficients a_1, a_2 were calculated. The approach of mathematicians differs considerably from that of physicists. It is very general but also very complicated and seems not to be well adopted to physical problems. It allowed to compute in addition the third coefficient a_3 Gilkey (1975).

It is the general manifestly covariant technique for the calculation of the coefficients a_k that was elaborated in my PhD thesis in Moscow University in 1987. This technique is essentially based on the use of the covariant Taylor expansions of all needed two-point quantities, such as the second derivatives of the geodesic interval and the first derivative of the parallel displacement operator. I solved the De Witt recursion relations explicitly and obtained a very effective nonrecursive covariant formulae for coefficients a_k. This method allowed me to calculate the coefficient a_4 as well as to maintain the previous results of other authors. These results are published in Avramidi (1989, 1990, 1991). The fourth coefficient a_4 for the case of scalar operators was also calculated by P. Amsterdamski, A. L. Berkin and D. J. O'Connor (1989).

Moreover, it allows to analyze the general structure of a_k and to calculate all of them in some approximation, for example, the terms with leading derivatives in all a_k.

One should mention here that the Schwinger - De Witt ansatz is actually an asymptotic expansion and not a Taylor series. This expansion does not converge, in general, namely in the case when the transfer function $\Omega(t)$ is not analytic at the point $t=0$.

5. New ansatz for the heat kernel

Therefore, I proposed in Avramidi (1990, 1991) a *new* ansatz for the transfer function $\Omega(t)$ which generalizes the Schwinger - De Witt one but is always valid. One can show that the transfer function can be always presented in the form of an inverse Mellin transform of a product of the Γ-function and an *entire* function b_q

$$\Omega(t)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} dq t^q\Gamma(-q)b_q \tag{43}$$

with $c < 0$. The function b_q must satisfy in addition some asymptotic conditions at $q \to \pm i\infty$ so that this integral converge at the infinity and one can simply move the contour of integration.

Then if we move the integration contour to the right, so that it cross the real axis between $N-1$ and N than the residues at simple poles of the Γ-function reproduce exactly the first N terms of the Schwinger - De Witt expansion

$$\Omega(t) = \sum_{k=0}^{N-1} \frac{(-t)^k}{k!} b_k + \frac{1}{2\pi i} \int\limits_{c_N - i\infty}^{c_N + i\infty} dq t^q \Gamma(-q) b_q \tag{44}$$

where $N - 1 < c_N < N$.

This formula is exact, in contrast to the original Schwinger - De Witt expansion. This means that it gives the exact form of the rest term of the Schwinger - De Witt expansion which can be evaluated from some independent grounds. Substituting this new ansatz in the equation for the transfer function we obtain the recursion relation for the function b_q

$$\left(1 + \frac{1}{q} D\right) b_q = \bar{F} b_{q-1} \tag{45}$$

with initial condition

$$b_0 = 1 \tag{46}$$

So, the function b_q:

1) contains the whole information about the heat kernel and, therefore, about the effective action,
2) is the analytic continuation of the HMDS-coefficients on the whole complex plane of their order, so that it is just b_k at integer positive points $q = k$,
3) it provides the evaluation of the rest term of the Schwinger - De Witt asymptotic expansion.

Thus the recursion equation for b_q is actually equivalent to the original heat equation. Using the function b_q we can get also a *new* ansatz for the complex power of the differential operator F

$$\begin{aligned} F^{-p} =& \frac{1}{\Gamma(p)} \int\limits_0^\infty dt\ t^{p-1} U(t) \\ =& (4\pi)^{-d/2} \frac{\Delta^{1/2}}{2\pi i \Gamma(p)} \int\limits_{c-i\infty}^{c+i\infty} dq \left(\frac{\sigma}{2}\right)^{p - \frac{d}{2} + q} \Gamma(-q) \Gamma\left(-p + \frac{d}{2} - q\right) b_q \end{aligned} \tag{47}$$

with $c < \frac{d}{2} - \mathrm{Re} p$. This form is especially useful in analyzing the short-distance behavior of the Green function F^{-1} when $x \to x'$ or $\sigma \to 0$.

Now taking the coincidence limit $x \to x'$ one can get the functional trace of the complex power of the differential operator F, i.e. the ζ-function

$$\zeta(p) = (4\pi)^{-d/2} \frac{\Gamma\left(p - \frac{d}{2}\right)}{\Gamma(p)} \mu^{2p} B_{\frac{d}{2}-p} \tag{48}$$

This is a very simple but very important formula. It displays the whole analytical structure of the ζ-function. Because B_q is an entire function it is immediately seen that $\zeta(p)$ is meromorphic function and all poles of it are given simply by the Γ-function. Anyway $\zeta(p)$ is analytic at $p = 0$ and one can calculate simply the values of ζ-function and its derivative at this point.

One has to distinguish here between the spaces of even and odd dimension. In even dimension $\zeta(0)$ is just the HMDS-coefficient of order $d/2$

$$\zeta(0) = (4\pi)^{-d/2} \frac{(-1)^{d/2}}{\Gamma(d/2+1)} B_{d/2}, \tag{49}$$

where

$$B_q = Trb_q = \int dx\ g^{1/2} \mathrm{tr} b_q \big|_{diag} \tag{50}$$

while in odd dimension it vanishes

$$\zeta(0) = 0. \tag{51}$$

The one-loop effective action is given by the formulas: for even dimension

$$\Gamma_{(1)} = \frac{1}{2}(4\pi)^{-d/2} \frac{(-1)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left\{ B'_{d/2} - \left[\log \mu^2 + \Psi\left(\frac{d}{2}+1\right) + C\right] B_{d/2} \right\} \tag{52}$$

where $\Psi(q) = (d/dq) \log \Gamma(q)$, $C = -\Psi(1)$ and

$$B'_{d/2} = \frac{d}{dq} B_q \bigg|_{q=d/2} \tag{53}$$

and for odd dimension

$$\Gamma_{(1)} = \frac{1}{2}(4\pi)^{-d/2} \frac{\pi(-1)^{\frac{d-1}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} B_{d/2} \tag{54}$$

The nontrivial contribution to the one-loop effective action is contained here in the first terms which do not depend on the renormalization ambiguity, i.e. on the renormparameter μ. These terms do not depend also on the regularization. The remaining term $B_{d/2} \log \mu$ (in even dimension) is just the renormalization one. We are free to add such terms to the effective action because they are different in different regularization schemes. The models that do not have such terms, i.e. when $\Gamma_{(1)}$ does not depend on the renormparameter μ are called one-loop finite. The examples of such models are: odd dimensional and sypersymmetric models. This does not mean, of course, that these models are completely

finite. To state this one has to prove independence on the renormparameter at all orders of the perturbation theory.

6. General structure of the asymptotic expansion

Now after I spent so much time to convince you how important it is to calculate the one-loop effective action, ζ-function and the heat kernel I am going to describe very briefly some methods for the calculation of these quantities and just present the results not going deeply into details.

Let me make some remarks on the subject.

- First of all, it is obviously impossible to evaluate the effective action exactly, even at the one-loop order. There are, of course, some simple special cases of background fields and geometries that allow the exact and even explicit calculation of the heat kernel or the effective action. However, the effective action is an *action*, i.e. a functional of background fields that should be varied to get the Green functions, the vacuum expectation values of various fields observables, such as energy-momentum tensor and Yang-Mills currents etc.. That is why one needs the effective action for *general* background and, therefore, one has to develop consistent *approximate* methods for its calculation.
- Second, in quantum gravity and gauge theories the effective action is a *covariant* functional, i.e. invariant under diffeomorphisms and gauge transformations. That is why the approximations for calculating the effective action have to preserve the *general covariance at each order*. The flat space perturbation theory is an example of bad approximation because it is not covariant.

6.1 Massive quantum fields in weak background fields

One of the most known and succesful covariant approximations is that of *weak background fields*, i.e. in the case when the Compton wave length of the massive field is much smaller than the characteristic length scale L of the background fields

$$\frac{\hbar}{mc} \ll L \tag{55}$$

In other words, all invariants build from the curvature and its covariant derivatives are much smaller than the corresponding power of the mass parameter

$$R \ll m^2, \qquad \nabla\nabla R \ll m^4, \dots \tag{56}$$

that means

$$a_k \ll m^{2k} \tag{57}$$

In this case one can simply use the Schwinger - De Witt ansatz for the heat kernel to get the $1/m^2$ - asymptotic expansion of the effective action: for odd d

$$\Gamma_{(1)} = \frac{1}{2}(4\pi)^{-d/2}\pi(-1)^{\frac{d-1}{2}}\sum_{k=0}^{\infty}\frac{m^{d-2k}}{k!\Gamma\left(\frac{d}{2}-k+1\right)}A_k \tag{58}$$

and for even dimension

$$\Gamma_{(1)} = \frac{1}{2}(4\pi)^{-d/2}\Bigg\{(-1)^{d/2}\sum_{k=0}^{d/2}\frac{m^{d-2k}}{k!\Gamma\left(\frac{d}{2}+1-k\right)}A_k\left[\ln\frac{m^2}{\mu^2}-\Psi\left(\frac{d}{2}-k+1\right)-C\right]$$
$$+\sum_{k=\frac{d}{2}+1}^{\infty}\frac{\Gamma\left(k-\frac{d}{2}\right)(-1)^k}{k!m^{2k-d}}A_k\Bigg\} \tag{59}$$

This is a good approximation in weak background fields and describes the physical effect of vacuum polarization of the massive fields. For example, in four-dimensional space this expansion looks like

$$\Gamma_{(1)} = \frac{1}{2}(4\pi)^{-2}\Bigg\{\frac{1}{2}m^4\left(\log\frac{m^2}{\mu^2}-\frac{3}{2}\right)A_0 + m^2\left(\log\frac{m^2}{\mu^2}-1\right)A_1 + \log\frac{m^2}{\mu^2}A_2$$
$$-\frac{1}{6m^2}A_3 + \frac{1}{24m^4}A_4 + O(\frac{1}{m^6})\Bigg\} \tag{60}$$

where A_0, A_1 and A_2 are the renormalization terms and $A_3 \sim R^3$ and $A_4 \sim R^4$ describe really the vacuum polarization effects. Variation of the $\Gamma_{(1)}$ with respect to the background metric gives then the vacuum expectation value of the energy-momentum tensor, variation with respect to the connection gives the vacuum ecpectation value of the Yang-Mills current etc.

However the Schwinger - De Witt approximation is of very limited applicability. It is not effective in the case of large or rapidly varying background fields and becomes meaningless in massless theories. Therefore, this approximation can not describe essentially nonperturbative nonlocal effects such as particle creation and the vacuum polarization by strong background fields.

6.2 General structure of HMDS-coefficients and partial summation

In fact, the effective action is a nonlocal and nonanalytical functional and posseses a sensible massless limit. But its calculation requires quite different methods. One such method which would exceed the limits of the Schwinger - De Witt asymptotic expansion is the *partial summation procedure* Vilkovisky (1984). It is based on the analysis of the general structure of the HMDS-coefficients. The HMDS-coefficients are the local polynomial invariants built from the curvature and its covariant derivatives. Therefore, one

can classify all the terms according to the number of curvatures and their derivatives. The terms with leading derivatives can be shown to have the followin structure $R\Box^{k-2}R$. Then it follows the class of terms with one more curvature etc. The last class of terms does not contain any covariant derivatives at all but only the powers of the curvature

$$A_k = \int dx g^{1/2} \mathrm{tr} \Bigg\{ R\Box^{k-2}R + \sum_{0\le i\le 2k-6} R\,\nabla^i R\,\nabla^{2k-6-i}R + \cdots + (\nabla R)R^{k-3}(\nabla R) + R^k \Bigg\} \tag{61}$$

Now one can try to sum up each class of terms separately to get the corresponding expansion of the heat kernel

$$\mathrm{Tr}U(t) = \int dx g^{1/2} (4\pi t)^{-d/2} \exp(-tm^2) \mathrm{tr} \Bigg\{ 1 - t\left(Q - \frac{1}{6}R\right) + t^2 R\chi(t\Box)R + \cdots + t^3 \nabla R \Psi(tR) \nabla R + \Phi(tR) \Bigg\} \tag{62}$$

and that of the effective action

$$\Gamma_{(1)} = \int dx g^{1/2} \mathrm{tr} \Bigg\{ RF(\Box)R + \cdots + \nabla R Z(R) \nabla R + V(R) \Bigg\} \tag{63}$$

We mention here once again that these expansions are asymptotic ones and do not converge, in general. So, one has to use some methods of summation of the divergent asymptotic series. This can be done by using an integral transform and analytic continuation.

Let us stress again that in quantum gravity and gauge theories the assumptions about the local behavior of the background fields must deal with physical gauge invariant properties of the local geometry, i.e. the curvature invariants but not with the behavior of the metric and the connection which is not invariant. Comparing the value of the curvature with that of its *covariant* derivatives one comes to two possible approximations.

7. High-energy approximation

There are here two general approximations. The first one is the so called high-energy approximation or the short-wave one. It is characterized by small but rapidly varying curvature. It means that the covariant derivatives of the curvatures are much greate than the powers of them

$$\nabla\nabla R \gg RR \tag{64}$$

Such a formulation is manifestly covariant and, therefore, more suitable for calculations in quantum gravity and gauge theories than the usual flat space perturbation theory. The terms with higher derivatives in HMDS-coefficients have the following form

$$A_k = \int dx\, g^{1/2} \mathrm{tr} \frac{(-1)^{k-2}\Gamma(k+1)\Gamma(k-1)}{2\Gamma(2k-2)} \Big\{ f_k^{(1)} Q \Box^{k-2} Q + 2 f_k^{(2)} \mathcal{R}_{\alpha\mu} \nabla^\alpha \Box^{k-3} \nabla_\nu \mathcal{R}^{\nu\mu} - 2 f_k^{(3)} Q \Box^{k-2} R$$
$$+ f_k^{(4)} R_{\mu\nu} \Box^{k-2} R^{\mu\nu} + f_k^{(5)} R \Box^{k-2} R + O(R^3) \Big\} \quad (65)$$

where the coefficients $f_k^{(i)}$ are given by

$$f_k^{(1)} = 1 \quad , \quad f_k^{(2)} = \frac{1}{2(2k-1)} \quad , \quad f_k^{(3)} = \frac{k-1}{2(2k-1)} \quad ,$$
$$f_k^{(4)} = \frac{1}{2(4k^2-1)} \quad , \quad f_k^{(5)} = \frac{k^2-k-1}{4(4k^2-1)} \quad (66)$$

They were calculated also in my PhD thesis in 1987 and published then in (1989, 1990). At the same time just the same result was obtained in T. Branson, P. B. Gilkey and B. Ørsted(1990) by using completely independent approach.

Now having *all* A_k and supposing that these are the main terms in the high-energy approximation we can try to sum up the local Schwinger - De Witt expansion to get the nonlocal heat kernel with appropriate formfactors

$$\mathrm{Tr}\, U(t) = \int dx\, g^{1/2} (4\pi t)^{-d/2} \exp(-tm^2) \mathrm{tr} \Big\{ 1 - t \left(Q - \frac{1}{6} R \right)$$
$$+ \frac{t^2}{2} \Big[Q \gamma^{(1)}(t\Box) Q + 2 \mathcal{R}_{\alpha\mu} \nabla^\alpha \frac{1}{\Box} \gamma^{(2)}(t\Box) \nabla_\nu \mathcal{R}^{\nu\mu} - 2 Q \gamma^{(3)}(t\Box) R$$
$$+ R_{\mu\nu} \gamma^{(4)}(t\Box) R^{\mu\nu} + R \gamma^{(5)}(t\Box) R \Big] + O(R^3) \Big\} \quad . \quad (67)$$

where the formfactors $\gamma^{(i)}(t\Box)$ have the form

$$\gamma^{(i)}(t\Box) = \int_0^1 d\xi\, f^{(i)}(\xi) \exp\left(\frac{1-\xi^2}{4} t\Box \right) \quad (68)$$

with some simple polynomial functions $f^{(i)}(\xi)$ given by

$$f^{(1)}(\xi) = 1 \quad , \quad f^{(2)}(\xi) = \frac{1}{2}\xi^2 \quad , \quad f^{(3)}(\xi) = \frac{1}{4}(1-\xi^2) \quad ,$$
$$f^{(4)}(\xi) = \frac{1}{6}\xi^4 \quad , \quad f^{(5)}(\xi) = \frac{1}{48}(3 - 6\xi^2 - \xi^4) \quad (69)$$

Using this heat kernel one obtains then the ζ-function and the effective action.

Another way to get the effective action that we have put forward is much more elegant and simple. One has first to calculate the coefficients B_k which are also *local* polynomials. They are just the HMDS-coefficients for non-vanishing mass. And then one has to make a very nontrivial trick, namely, the *analytic continuation* of these coefficients. In this way we obtain the function B_q without solving the recursion relations. Of course, we have used them to calculate the HMDS-coefficients A_k.

Having the analytic function B_q one can calculate then very simply the ζ-function and the effective action.

We stress here again that the heat kernel as well as the function B_q (for noninteger q) and ζ-function and effective action are *nonlocal* functionals of the background fields. The result for the effective action looks like Avramidi (1989, 1990)

$$\Gamma_{(1)} = \Gamma_{(1)loc} + \Gamma_{(1)nonloc} \tag{70}$$

Here the local part is equal: in odd d

$$\Gamma_{(1)loc} = \frac{1}{2}(4\pi)^{-d/2}\frac{\pi(-1)^{\frac{d-1}{2}}}{\Gamma(\frac{d}{2}+1)}\int dx\, g^{1/2} str\left\{m^d + \frac{d}{2}m^{d-2}\left(Q - \frac{1}{6}R\right) + O(R^3)\right\} \tag{71}$$

and in even dimension d

$$\begin{aligned}\Gamma_{(1)loc} = &\frac{1}{2}(4\pi)^{-d/2}\frac{(-1)^{d/2}}{\Gamma(\frac{d}{2}+1)}\int dx\, g^{1/2} str\Bigg\{m^d\left[\ln\frac{m^2}{\mu^2} - \Psi\left(\frac{d}{2}\right) - \mathbf{C}\right] \\ &+ \frac{d}{2}m^{d-2}\left[\ln\frac{m^2}{\mu^2} - \Psi\left(\frac{d}{2}\right) - \mathbf{C}\right]\left(Q - \frac{1}{6}R\right) + O(R^3)\Bigg\} \quad .\end{aligned} \tag{72}$$

The nonlocal part of the effective action can be written down in the form

$$\begin{aligned}\Gamma_{(1)nonloc} = &\frac{1}{2}(4\pi)^{-d/2}\int dx\, g^{1/2}\, str\Bigg\{Q\beta^{(1)}(\Box)Q + 2\mathcal{R}_{\alpha\mu}\nabla^{\alpha}\frac{1}{\Box}\beta^{(2)}(\Box)\nabla_{\nu}\mathcal{R}^{\nu\mu} \\ &- 2Q\beta^{(3)}(\Box)R + R_{\mu\nu}\beta^{(4)}(\Box)R^{\mu\nu} + R\beta^{(5)}(\Box)R + O(R^3)\Bigg\}\end{aligned} \tag{73}$$

where $\beta^{(i)}(\Box)$ are *nonlocal* formfactors. They have the following integral representation: for odd d

$$\beta^{(i)}(\Box) = \frac{\pi(-1)^{(d-1)/2}}{2\Gamma(\frac{d}{2}-1)}\int_0^1 d\xi\, f^{(i)}(\xi)\left(m^2 - \frac{1-\xi^2}{4}\Box\right)^{\frac{d}{2}-2} \tag{74}$$

and for even d

$$\begin{aligned}\beta^{(i)}(\Box) = &\frac{(-1)^{d/2}}{2\Gamma(\frac{d}{2}-1)}\int_0^1 d\xi\, f^{(i)}(\xi)\left(m^2 - \frac{1-\xi^2}{4}\Box\right)^{\frac{d}{2}-2} \\ &\times\left\{\ln\left[\frac{1}{\mu^2}\left(m^2 - \frac{1-\xi^2}{4}\Box\right)\right] - \Psi\left(\frac{d}{2}-1\right) - C\right\} \quad .\end{aligned} \tag{75}$$

with the same functions $f^{(i)}$.

Here it is immediately seen the difference between the even and odd dimension. The formfactors in odd dimension do not have the logarithmic term depending on the renorm-parameter. This is a direct consequence of the ultraviolet finiteness of the effective action in this case.

One can show that this integrals define analytic functions on the whole complex plane with the cut along the positive real ais from $m^2/4$ to ∞. The asymptotic of the formfactors at the infinity have the form

$$\beta(\lambda)\Big|_{\lambda\to\infty} = (-\lambda)^{d/2-2}\left(C_1 \log\frac{-\lambda}{\mu^2} + C_2 + O(\frac{m^2}{\lambda})\right) \tag{76}$$

Moreover, using this integral representation one can analyse their analytic properties, calculate their high-energy limits and imaginary parts in the pseudo-Euclidean region above the threshold etc..

Let us consider a simple example illustrating our technique, namely the massless conform invariant scalar field on a two-dimensional manifold. In this case the HMDS-coefficients have a very simple explicit form

$$B_k = \frac{k(k-1)}{2}\frac{(\Gamma(k+1))^2}{\Gamma(2k+2)}\int dx g^{1/2} R \Box^{k-2} R \tag{77}$$

This formula gives immediately the function B_q simply by putting k to be complex. Calculating then the derivative of the function B_q at the point $q=1$ we obtain the corresponding effective action.

$$\Gamma_{(1)} = -\frac{1}{2(4\pi)}B_1' = \frac{1}{24(4\pi)}\int dx\, g^{1/2}\left\{R\frac{1}{\Box}R + O(R^3)\right\} \tag{78}$$

This is exactly the famous Polyakov (1981) effective action which was obtained by using completely different method, namely, by integrating the conformal anomaly. The point is the following. Any two-dimensional manifold is conformally flat. Therefore, any functional of the metric can be determined by the functional derivative with respect to conformal factor, i.e. by the so called trace anomaly.

One has to mention here the work of Barvinsky and Vilkovisky (1987, 1990) and Barvinsky, Gusev, Zhitnikov and Vilkovisky (1993) who developed a different method for calculating the nonlocal effective action. They managed to calculate the next *third order* in the nonlocal curvature expansion.

8. Low-energy approximation

Let us go over now to the opposite *low-energy* or long-wave approximation, which is used for calculating the effective potential in quantum field theory. The low-energy effective action is determined by strong slowly varying background fields. Therefore, it can be obtained by summing up the terms without derivatives in the first place. It leads to a

local but nonanalytical functional. Such a result can not be obtained by any perturbation theory and is essentially nonperturbative.

This means that the derivatives of the curvature are much smaller than the powers of it.

$$\nabla\nabla R \ll RR \tag{79}$$

To obtain the effective potential it is sufficent to consider the lowest (zeroth) order of the low-energy approximation. It means that we can put all covariant derivatives of the curvature and the potential term equal to zero

$$\nabla_\mu R_{\alpha\beta\gamma\delta} = 0, \qquad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0, \qquad \nabla_\mu Q = 0. \tag{80}$$

Mention that these conditions are local. The determine the geometry of the symmetric spaces. They can be of very different topological structure. However, we do not investigate the influence of the topology and concentrate our attention on the local effects.

In physical problems the correct setting of the problem is as follows. Consider a complete noncompact asymptotically flat manifold without boundary that is homeomorphic to $\mathbb{R}^{\rm d}$. Let a finite region of the manifold exist that is strongly curved and quasi-homogenous, i.e. the local invariants of the curvature vary very slowly. Then the geometry of this region is locally very similar to that of a symmetric space. This can be imagined as follows. Take the flat Euclidean space $\mathbb{R}^{\rm d}$, cut out from it a region M with some boundary ∂M and stick to it along the boundary, instead of the piece cut out, a piece of a curved symmetric space with the same boundary ∂M.

This case is more comlicated than the high-energy one. Nevertheless, the problem turns out to be purely algebraic one. I will briefly present here only the main idea and the results obtained in Avramidi (1993, 1994).

8.1 Heat kernel in flat space with nonvanishing Yang-Mills background

So, let us consider first the case of Yang-Mills background fields and flat space, i.e. the Riemann curvature vanishes

$$R_{\alpha\beta\gamma\delta} = 0 \tag{81}$$

The operators of covariant derivatives toghether with the curvature build a nilpotent Lie algebra

$$[\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu} \tag{82}$$

$$[\nabla_\mu, \mathcal{R}_{\alpha\beta}] = [\mathcal{R}_{\mu\nu}, \mathcal{R}_{\alpha\beta}] = [\mathcal{R}_{\mu\nu}, Q] = 0 \tag{83}$$

One can prove an algebraic theorem that expresses the heat kernel operator, i.e. the exponential of a second order operator, in terms of the group elements, i.e. the exponential of the first order operators.

$$\begin{aligned}\exp(t\,\Box) =& (4\pi t)^{-d/2} \det\left(\frac{tg^{-1}\mathcal{R}}{\sinh(tg^{-1}\mathcal{R})}\right)^{1/2} \\ & \int dk \exp\left\{-\frac{1}{4t} k^\mu (t\mathcal{R}\coth(g^{-1}t\mathcal{R}))_{\mu\nu} k^\nu + k^\mu \nabla_\mu\right\}\end{aligned} \tag{84}$$

Using this representation one can act on the δ-function and obtain the trace of the heat kernel

$$\mathrm{Tr}U(t) = \int dx g^{1/2}(4\pi t)^{-d/2}\mathrm{tr}\left\{\exp\left(-t(m^2+Q)\right)\det\left(\frac{tg^{-1}\mathcal{R}}{\sinh(tg^{-1}\mathcal{R})}\right)^{1/2}\right\} \tag{85}$$

and, therefore, the ζ-function and the corresponding effective action. This is the generalization of the well-known Schwinger's result in quantum electrodynamics, i.e. in the case of Abelian $U(1)$ gauge group. Here the determinant is taken over the spacetime indices and the trace over the group ones.

This is a good example how one can get the heat kernel without solving any differential equations.

8.2 Heat kernel for a scalar field in symmetric space

The second example is more complicated and interesting. This is just a scalar field in symmetric space, that means that the curvature $\mathcal{R}_{\mu\nu}$ is equal to zero

$$\mathcal{R}_{\mu\nu} = 0 \tag{86}$$

Our main idea remains the same - to express the heat kernel operator in terms of some exponentials of first order operators.

Let us express first the Riemann curvature of the symmetric space in the form

$$R^a{}_b{}^c{}_d = \beta^{ik} D^a{}_{ib} D^c{}_{kd} \tag{87}$$

Here the matrices $D_i = \{D^a{}_{ib}\}$ are the generators of the isotropy algebra

$$[D_i, D_k] = F^j{}_{ik} D_j \tag{88}$$

and $F^j{}_{ik}$ are the structure constants of it. Matrix β^{ik} is some symmetric nondegenerate matrix. It can be treated as the metric in the isotropy group and can be used to raise and lower the indices of the isotropy algebra.

Then one can introduce the constants $C^A{}_{BC} = -C^A{}_{CB}$

$$C^i{}_{ab} = E^i{}_{ab}, \quad C^a{}_{ib} = D^a{}_{ib}, \quad C^i{}_{kl} = F^i{}_{kl}, \tag{89}$$
$$C^a{}_{bc} = C^i{}_{ka} = C^a{}_{ik} = 0,$$

and show that they satisfy the Jacobi identities

$$C^E{}_{D[A} C^D{}_{BC]} = 0 \tag{90}$$

This means that these constants are the structure constants of some Lie algebra. This is exactly the Lie algebra of the infinitesimal isometries

$$[\xi_A, \xi_B] = C^C{}_{AB} \xi_C \tag{91}$$

Further, introducing the Cartan metric on this algebra

$$\gamma_{AB} = \begin{pmatrix} g_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix}. \tag{92}$$

one can present the Laplacian in terms of the generators of the isometries

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = \gamma^{AB}\xi_A\xi_B \tag{93}$$

Using this representation and the structure of the algebra of isometries one can prove a theorem, expressing the heat kernel operator in terms of an average over the group of isometries

$$\exp(t\,\Box) = (4\pi t)^{-D/2}\int dk\gamma^{1/2}\det\left(\frac{\sinh(k^A C_A/2)}{k^A C_A/2}\right)^{1/2} \times \exp\left\{-\frac{1}{4t}k^A\gamma_{AB}k^B + \frac{1}{6}R_G t\right\}\exp(k^A\xi_A), \tag{94}$$

Here

$$R_G = -\frac{1}{4}\gamma^{AB}C^C{}_{AD}C^D{}_{BC} \tag{95}$$

is the scalar curvature of the group manifold and the matrices $C_A = \{C^B{}_{AC}\}$ and $F_i = \{F^j{}_{ik}\}$ are the generators of adjoint representations of the algebra of isometries and the isotropy subalgebra.

Now acting with this heat kernel on the δ-function, one can obtain finally the trace of the heat kernel

$$\mathrm{Tr}U(t) = \int dx g^{1/2}(4\pi t)^{-d/2}\exp\left\{-t\left(m^2 + Q - \frac{1}{6}R_G\right)\right\} \times\left\langle \det\left(\frac{\sinh(\sqrt{t}\omega^i F_i/2)}{\sqrt{t}\omega^i F_i/2}\right)^{1/2}\det\left(\frac{\sinh(\sqrt{t}\omega^i D_i/2)}{\sqrt{t}\omega^i D_i/2}\right)^{-1/2}\right\rangle \tag{96}$$

in form of an Gaussian average over the isotropy subgroup

$$< f(\omega) >= (4\pi)^{-p/2}\int d\omega\beta^{1/2}\exp\left(-\frac{1}{4}\omega^i\beta_{ik}\omega^k\right)f(\omega) \tag{97}$$

This expression is manifestly covariant, because all its ingredients, matrices F_i, D_i and β_{ik} are the invariants of the curvature tensor. If we expand this heat kernel in asymptotic series in powers of t then we recover *all* HMDS-coefficients for *all* symmetric spaces. They will be expressed then in terms of various foldings of the quantities F_i, D_i and β_{ik}. But these are just the curvature invariants. That means that one can obtain the explicit formulae for HMDS-coefficients in terms of the curvature.

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